Chapter 5

Fuzzy Logic Control for Business, Finance, and Management

Fuzzy logic control methodology has been developed mainly for the needs of industrial engineering. This chapter introduces the basic architecture of fuzzy logic control for the needs of business, finance, and management. It will show how decisions can be made by using and aggregating *if ... then* inferential rules. Instead of trying to build conventional mathematical models, a task almost impossible when complex phenomena are under study, the presented methodology creats fuzzy logic models reflecting a given situation in reality and provides solution leading to suggestion for action. Application is made to a client financial risk tolerance ability model.

5.1 Introduction

Complex systems involve various types of fuzziness and undoubtedly represent an enormous challenge to the modelers.

The classical control methodologies developed mainly for engineering are usually based on mathematical models of the objects to be controlled. Mathematical models simplify and conceptualize events in nature and human activities by employing various types of equations which must be solved. However, the use of mathematical models gives rise to the question how accurate they reflect reality. In complicated cases the construction of such models might be impossible. This is especially true for business, financial, and managerial systems which involve a great number of interacting factors, some of socio-psychological nature.

Fuzzy logic models employ fuzzy sets to handle and describe imprecise and complex phenomena and uses logic operations to arrive to conclusion.

Fuzzy sets (in particular fuzzy numbers) and fuzzy logic applied to control problems form a field of knowledge called *fuzzy logic control* (FLC).[1] It deals with control problems in an environment of uncertainty and imprecision; it is very effective when high precision is not required and the control object has variables available for measurement or estimation.

Imitating human judgment in common sense reasoning FLC uses linguistic values framed in *if ... then* rules. For instance: *if client's annual income is low and total networth is high*, *then client's risk tolerance is moderate*. Here the linguistic variables *annual income* and *total networth* are *inputs*; the linguistic variable *risk tolerance* is *output*; *low*, *high*, and *moderate* are *values* (*terms* or *labels*) of linguistic variables.

The implementation of FLC requires the development of a *knowledge base* which would make possible the stipulation of *if ... then* rules by using fuzzy sets. Important role here plays the experience and knowledge of human experts. They should be able to state the objective of the system to be controlled.

The goal of control in engineering is action. In business, finance, and management we expand the meaning of control and give broader interpretation of action; it might be also advise, suggestion, instruction, conclusion, evaluation, forecasting.

This chapter introduces the basic architecture of FLC. It shows how control problems can be solved by *if ... then* inferential rules without using conventional mathematical models. The presented methodology of heuristic nature can be easily applied to numerous control problems in industry, business, finance, and management. FLC is effective when a good solution is sought; it cannot be used to find the optimal (best) solution. However in the real world it is difficult to determine what is meant by the best.

A block diagram for control processes is depicted in Fig. 5.1. The meaning of each block is explained in the sections in this chapter.

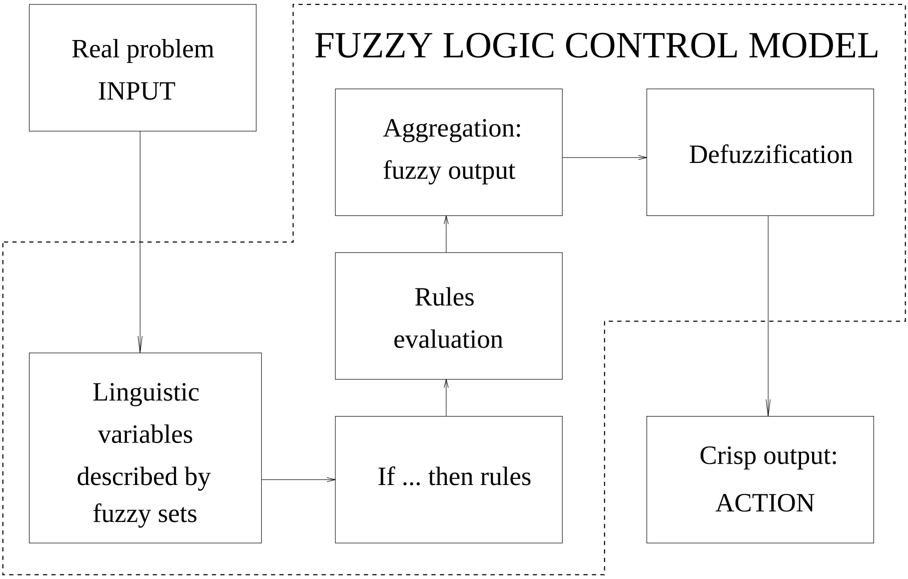

Fig. 5.1. Block diagram for fuzzy logic control process.

The FLC process will be illustrated step by step on a simplified *client financial risk tolerance model.*

5.2 Modeling the Control Variables

Control problems have *inputs* and *outputs* considered to be *linguistic variables.*

Here we explain the FLC technique on a system with two inputs $\mathcal{A}, \mathcal{B}$ and one output $\mathcal{C}$. The same technique can be extended and applied to problems with more inputs and outputs. It can be applied also in the case when the problem has only one input and one output.

Linguistic variables are modeled by sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ (see Section 2.4) containing certain number of terms $\mathcal{A}_i, \mathcal{B}_j, \mathcal{C}_k$:

$$\begin{aligned} \mathcal{A} &= \{\mathcal{A}_1, \dots, \mathcal{A}_i, \mathcal{A}_{i+1}, \dots, \mathcal{A}_n\}, \\ \mathcal{B} &= \{\mathcal{B}_1, \dots, \mathcal{B}_j, \mathcal{B}_{j+1}, \dots, \mathcal{B}_m\}, \\ \mathcal{C} &= \{\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_l\}. \end{aligned} \tag{5.1}$$

The terms $\mathcal{A}_i, \mathcal{B}_j$, and $\mathcal{C}_k$ are fuzzy sets defined as

$$\begin{aligned} \mathcal{A}_i &= \{(x, \mu_{\mathcal{A}_i}(x)) | x \in A_i \subset U_1\}, \quad i = 1, \dots, n, \\ \mathcal{B}_j &= \{(y, \mu_{\mathcal{B}_j}(y)) | y \in B_j \subset U_2\}, \quad j = 1, \dots, m, \\ \mathcal{C}_k &= \{(z, \mu_{\mathcal{C}_k}(z)) | z \in C_k \subset U_3\}, \quad k = 1, \dots, l. \end{aligned} \tag{5.2}$$

The design of the sets (5.2) requires:

(i) Determination of the universal sets U_1, U_2, U_3 (or operating domains) of the base variables x, y, z for the linguistic variables described by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ (see Section 2.4).

(ii) Selection of shapes, peaks, and flats of the membership functions of $\mathcal{A}_i, \mathcal{B}_j, \mathcal{C}_k$ (the terms). Most often triangular, trapezoidal, or bell-shaped types of membership functions are used (or part of these), hence then (5.2) are fuzzy numbers.

(iii) Specifying the number of terms in (5.1), i.e. the numbers n, m, and l. Usually these numbers are between 2 and 7.

(iv) Specifying the supporting intervals (domains) for the terms $\mathcal{A}_i, \mathcal{B}_j, \mathcal{C}_k$.

Case Study 17 (Part 1) *A Client Financial Risk Tolerance Model*

Financial service institutions face a difficult task in evaluating clients risk tolerance. It is a major component for the design of an investment policy and understanding the implication of possible investment options in terms of safety and suitability.

Here we present a simple model of client's *risk tolerance ability* which depends on his/hers *annual income* (*AI*) and *total networth* (*TNW*).

The control objective of the client financial risk tolerance policy model is for any given pair of input variables (*annual income*, *total networth*) to find a corresponding output, a *risk tolerance* (*RT*) level.

Suppose the financial experts agree to describe the input variables *annual income* and total *networth* and the output variable *risk tolerance* by the sets (particular case of (5.1)):

$$\begin{aligned} \textit{Annual invcome} &\triangleq \mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = \{\mathbf{L}, \mathbf{M}, \mathbf{H}\}, \\ \textit{Total networth} &\triangleq \mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\} = \{\mathbf{L}, \mathbf{M}, \mathbf{H}\}, \\ \textit{Risk tolerance} &\triangleq \mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\} = \{\mathbf{L}, \mathbf{MO}, \mathbf{H}\}, \end{aligned}$$

hence the number of terms in each term set is $n = m = l = 3$. The terms have the following meaning: $\mathbf{L} \triangleq \textit{low}, \mathbf{M} \triangleq \textit{medium}, \mathbf{H} \triangleq \textit{high}$, and $\mathbf{MO} \triangleq \textit{moderate}$. They are fuzzy numbers whose supporting intervals belong to the universal sets $U_1 = \{x \times 10^3 | 0 \le x \le 100\}, U_2 = \{y \times 10^4 | 0 \le y \le 100\}, U_3 = \{z | 0 \le z \le 100\}$ (see Figs. 5.2–5.4). The real numbers x and y represent dollars in thousands and hundred of thousands, correspondingly, while z takes values on a psychometric scale from 0 to 100 measuring risk tolerance. The numbers on that scale have specified meaning for the financial experts.

The terms of the linguistic variables *annual income, total networth*, and *risk tolerance* described by triangular and part of trapezoidal numbers formally have the same membership functions presented analytically below (see (1.13) and (1.15)):

$$\begin{aligned} \mu_L(v) &= \begin{cases} 1 & \text{for} \quad 0 \le v \le 20, \\ \frac{50-v}{30} & \text{for} \quad 20 \le v \le 50, \end{cases} \\ \mu_M(v) &= \begin{cases} \frac{v-20}{30} & \text{for} \quad 20 \le v \le 50, \\ \frac{80-v}{30} & \text{for} \quad 50 \le v \le 80, \end{cases} \\ \mu_H(v) &= \begin{cases} \frac{v-50}{30} & \text{for} \quad 50 \le v \le 80, \\ 1 & \text{for} \quad 80 \le v \le 100. \end{cases} \end{aligned} \tag{5.3}$$

Here v stands for x, y, and z, meaning x substituted for v in (5.3) gives the equations of the terms in Fig. 5.2, y substituted for v produces the equations of terms in Fig. 5.3, and z substituted for v gives the equations of the terms in Fig. 5.4 (the second term $\mu_M(v)$ should read $\mu_{MO}(z)$).

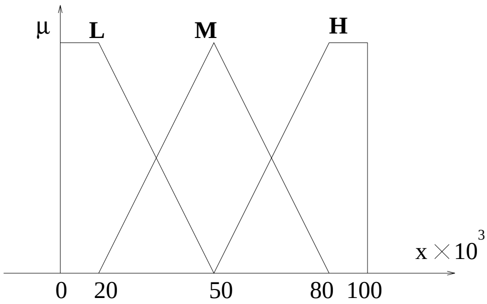

Fig. 5.2. Terms of the input *annual income*.

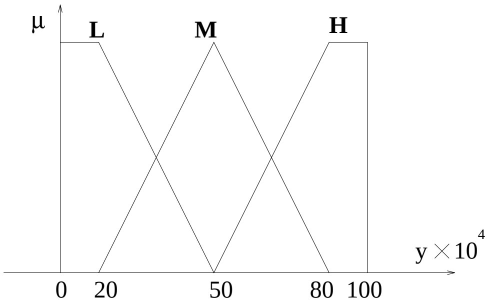

Fig. 5.3. Terms of the input *total networth*.

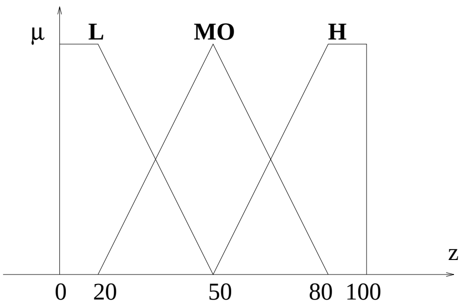

Fig. 5.4. Terms of the output *risk tolerance*.

□

5.3 If ... and ... then Rules

Next step is setting the *if ... and ... then* rules of inference called also *control rules* or *production rules.*

The number of the rules is nm, the product of the number of terms in each input linguistic variable $\mathcal{A}$ and $\mathcal{B}$ (see (5.1)).[2] The rules are designed to produce or have as a conclusion or consequence $l < nm$ different outputs (l is the number of terms in the output variable $\mathcal{C}$).

The rules with the possible fuzzy outputs labeled $\mathcal{C}_{ij}$ are presented symbolically on the rectangular $n \times m$ (n rows and m columns) Table 5.1 called decision table where $\mathcal{C}_{ij}, i = 1, \ldots, n, \quad j = 1, \ldots, m$, are renamed elements of the set $\{\mathcal{C}_1, \ldots, \mathcal{C}_l\}$.

Table 5.1. Decision table: *if ... and ... then* rules.

	$\mathcal{B}_1$	$\cdots$	$\mathcal{B}_j$	$\mathcal{B}_{j+1}$	$\cdots$	$\mathcal{B}_m$
$\mathcal{A}_1$	$\mathcal{C}_{11}$	$\cdots$	$\mathcal{C}_{1j}$	$\mathcal{C}_{1,j+1}$	$\cdots$	$\mathcal{C}_{1,m}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mathcal{A}_i$	$\mathcal{C}_{i1}$	$\cdots$	$\mathcal{C}_{ij}$	$\mathcal{C}_{i,j+1}$	$\cdots$	$\mathcal{C}_{i,m}$
$\mathcal{A}_{i+1}$	$\mathcal{C}_{i+1,1}$	$\cdots$	$\mathcal{C}_{i+1,j}$	$\mathcal{C}_{i+1,j+1}$	$\cdots$	$\mathcal{C}_{i+1,m}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mathcal{A}_n$	$\mathcal{C}_{n1}$	$\cdots$	$\mathcal{C}_{nj}$	$\mathcal{C}_{n,j+1}$	$\cdots$	$\mathcal{C}_{nm}$

The actual meaning of the *if ... and ... then* rules is

$$\textit{If } x \textit{ is } \mathcal{A}_i \textit{ and } y \textit{ is } \mathcal{B}_j \quad \textit{then } z \textit{ is } \mathcal{C}_k. \tag{5.4}$$

On Table 5.1, $\mathcal{C}_k$ renamed $\mathcal{C}_{ij}$ is located in the cell at the intersection of ith row and jth column. Denoting

$$p_i \triangleq x \textit{ is } \mathcal{A}_i, \quad q_j \triangleq y \textit{ is } \mathcal{B}_j, \quad r_k \triangleq z \textit{ is } \mathcal{C}_k, \tag{5.5}$$

we can write (5.4) as

$$\textit{If } p_i \textit{ and } q_j \textit{ then } r_k, \; r_k = r_{ij}. \tag{5.6}$$

The *and* part in (5.4) and (5.6), called *precondition*,

$$x \textit{ is } \mathcal{A}_i \text{ and } y \textit{ is } \mathcal{B}_j, \text{ i.e. } p_i \textit{ and } q_j, \tag{5.7}$$

is defined to be *composition conjunction* (2.10). It is a fuzzy relation in $A \times B \subseteq U_1 \times U_2$ with membership function

$$p_i \wedge q_j = \min(\mu_{\mathcal{A}_i}(x), \mu_{\mathcal{B}_j}(y)), \quad (x, y) \in A \times B \subset U_1 \times U_2. \tag{5.8}$$

The *if ... then* rule of inference (5.6) is implication. It expresses the truth of the precondition. There are several ways to define this rule. Here following Mamdani (1975) we define the *rule of inference* as a *conjunction-based* rule expressed by operation $\wedge$(min); r_k is called *conclusion* or *consequent*. Hence (5.6) can be presented in the form

$$p_i \wedge q_j \wedge r_k = \min(\mu_{\mathcal{A}_i}(x), \mu_{\mathcal{B}_j}(y), \mu_{\mathcal{C}_{ij}}(z)), \;\; r_k = r_{ij}, \tag{5.9}$$

$i = 1, \ldots, n; j = 1, \ldots, m; k = 1, \ldots, l$; and $(x, y, z) \in A \times B \times C \subseteq U_1 \times U_2 \times U_3$.

This presentation gives the truth value of the rule which is the result of the min operation on the membership functions of the fuzzy sets $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.

Case Study 17 (Part 2) *A Client Financial Risk Tolerance Model*

For the client financial risk tolerance model in Case Study 17 (Part 1), $n = m = l = 3$. Hence the number of *if ... then* rules is 9 and the number of different outputs is 3. Assume that the financial experts selected the rules presented on the decision Table 5.2.

Table 5.2. *If ... and ... then* rules for the client financial risk tolerance model.

Total networth $\mathcal{B} \longrightarrow$

Annual income $\mathcal{A} \downarrow$

	L	**M**	**H**
L	**L**	**L**	**MO**
M	**L**	**MO**	**H**
H	**MO**	**H**	**H**

The rules have as a conclusion the terms in the output $\mathcal{C}$ (see 5.3). They read:

Rule 1: *If client's annual income* (*CAI*) *is low* (**L**) *and client's total networth* (*CTN*) *is low* (**L**), *then client's risk tolerance* (*CRT*) *is low* (**L**);

Rule 2: *If CAI is* **L** *and CTN is medium* (**M**), *then CRT is* **L**;
Rule 3: *If CAI is* **L** *and CTN is high* (**H**), *then CRT is moderate* (**MO**);
Rule 4: *If CAI is* **M** *and CTN is* **L**, *then CRT is* **L**;
Rule 5: *If CAI is* **M** *and CTN is* **M**, *then CRT is* **MO**;
Rule 6: *If CAI is* **M** *and CTN is* **H**, *then CRT is* **H**;
Rule 7: *If CAI is* **H** *and CTN is* **L**, *then CRT is* **MO**;
Rule 8: *If CAI is* **H** *and CTN is* **M**, *then CRT is* **H**;
Rule 9: *If CAI is* **H** *and CTN is* **H**, *then CRT is* **H**.

Using the notations (5.5)–(5.8) the above rules can be presented in the form (5.9):

$$\begin{aligned}
&\text{Rule 1:} \quad p_1 \wedge q_1 \wedge r_{11} &&= \min(\mu_{\mathbf{L}}(x), \mu_{\mathbf{L}}(y), \mu_{\mathbf{L}}(z)),\\
&\text{Rule 2:} \quad p_1 \wedge q_2 \wedge r_{12} &&= \min(\mu_{\mathbf{L}}(x), \mu_{\mathbf{M}}(y), \mu_{\mathbf{L}}(z)),\\
&\text{Rule 3:} \quad p_1 \wedge q_3 \wedge r_{13} &&= \min(\mu_{\mathbf{L}}(x), \mu_{\mathbf{H}}(y), \mu_{\mathbf{MO}}(z)),\\
&\text{Rule 4:} \quad p_2 \wedge q_1 \wedge r_{21} &&= \min(\mu_{\mathbf{M}}(x), \mu_{\mathbf{L}}(y), \mu_{\mathbf{L}}(z)),\\
&\text{Rule 5:} \quad p_2 \wedge q_2 \wedge r_{23} &&= \min(\mu_{\mathbf{M}}(x), \mu_{\mathbf{M}}(y), \mu_{\mathbf{MO}}(z)),\\
&\text{Rule 6:} \quad p_2 \wedge q_3 \wedge r_{23} &&= \min(\mu_{\mathbf{M}}(x), \mu_{\mathbf{H}}(y), \mu_{\mathbf{H}}(z)),\\
&\text{Rule 7:} \quad p_3 \wedge q_1 \wedge r_{31} &&= \min(\mu_{\mathbf{H}}(x), \mu_{\mathbf{L}}(y), \mu_{\mathbf{MO}}(z)),\\
&\text{Rule 8:} \quad p_3 \wedge q_2 \wedge r_{32} &&= \min(\mu_{\mathbf{H}}(x), \mu_{\mathbf{M}}(y), \mu_{\mathbf{H}}(z)),\\
&\text{Rule 9:} \quad p_3 \wedge q_3 \wedge r_{33} &&= \min(\mu_{\mathbf{H}}(x), \mu_{\mathbf{M}}(y), \mu_{\mathbf{H}}(z)).
\end{aligned}$$

These rules stem from everyday life. It is quite natural for a person with low income and low networth to undertake a low risk and a person with high annual income and high networth to afford high risk. However, for various reasons a client may not want to tolerate high risk or on the contrary, may be willing to accept it regardless of income and networth. The experts, following a discussion with the client eventually have to redesign the rules. For instance, in the first case when the client prefers not to take a high risk, the conclusion part of the rules could be changed: in rules 3, 5, and 7, **MO** could be substituted by **L**; in rules 6 and 8, **H** could be substituted by **MO**. That will ensure a lower risk tolerance for the client which will lead to a more conservative investment policy.

□

5.4 Rule Evaluation

If the inputs to the FLC model are $x = x_0$ and $y = y_0$, then we have to find a corresponding value of the output z. The real numbers x_0 and y_0 are called readings; they can be obtained by measurement, observation, estimation, etc. To enter the FLC model, x_0 and y_0 have to be translated to proper terms of the corresponding linguistic variables.

A reading has to be matched against the appropriate membership functions representing terms of the linguistic variable. The matching is necessary because of the overlapping of terms (see Figs. 5.2, 5.3); it is called *coding the inputs.*

This is illustrated in Fig. 5.5 where to the reading $x_0 \in U_1$ there correspond two constant values, $\mu_{\mathcal{A}_i}(x_0)$ and $\mu_{\mathcal{A}_{i+1}}(x_0)$ called *fuzzy reading inputs.* They can be interpreted as the truth values of x_0 related to $\mathcal{A}_i$ and to $\mathcal{A}_{i+1}$, correspondingly.

In the same way we can obtain the *fuzzy reading inputs* corresponding to the reading $y_0 \in U_2$ (Fig. 5.6). In both figures only several terms of the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ (see (5.1)) are presented.

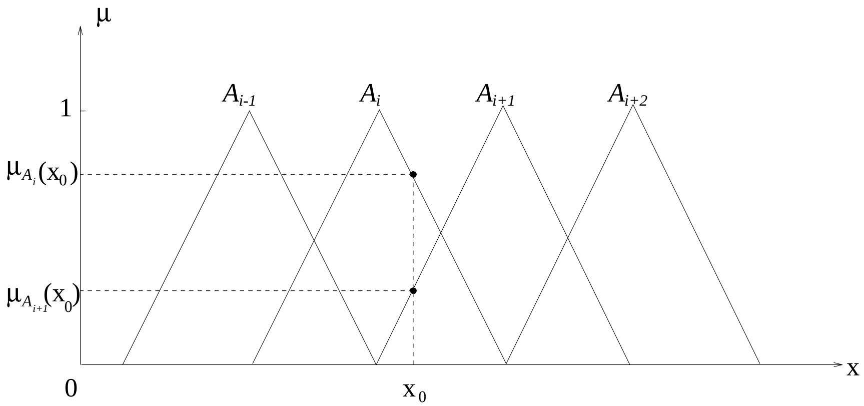

Fig. 5.5. Fuzzy reading inputs corresponding to reading x_0.

The straight line passing through x_0 parallel to μ axis intersects only the terms $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ of $\mathcal{A}$ in (5.1) thus reducing the fuzzy terms to crisp values (singletons) denoted $\mu_{\mathcal{A}_i}(x_0), \mu_{\mathcal{A}_{i+1}}(x_0)$. The line $x = x_0$ does not intersect the rest of the terms, hence we may say that the intersection is empty set with membership function 0. Similarly the line passing through y_0 intersects only the terms $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ of $\mathcal{B}$ in (5.1) which gives the crisp values (singletons) $\mu_{\mathcal{B}_j}(y_0), \mu_{\mathcal{B}_{j+1}}(y_0)$.

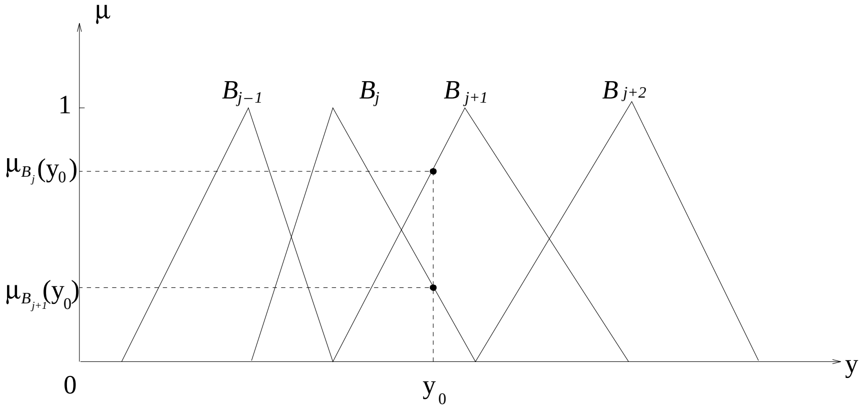

Fig. 5.6. Fuzzy reading inputs corresponding to reading y_0.

The decision Table 5.1 with $x = x_0$ and $y = y_0$, and the terms substituted by their corresponding membership functions, reduces to Table 5.3 which we call *induced decision table.*

Table 5.3. Induced decision table and active cells.

	0	$\cdots$	$\mu_{\mathcal{B}_j}(y_0)$	$\mu_{\mathcal{B}_{j+1}}(y_0)$	$\cdots$	0
0	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mu_{\mathcal{A}_i}(x_0)$	0	$\cdots$	$\mu_{\mathcal{C}_{ij}}(z)$	$\mu_{\mathcal{C}_{i,j+1}}(z)$	$\cdots$	0
$\mu_{\mathcal{A}_{i+1}}(x_0$	0	$\cdots$	$\mu_{\mathcal{C}_{i+1,j}}(z)$	$\mu_{\mathcal{C}_{i+1,j+1}}(z)$	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
0	0	$\cdots$	0	0	$\cdots$	0

Only four cells contain nonzero terms. Let us call these cells *active.* This can be seen from rules (5.8); if for $x = x_0$ and $y = y_0$ at least one of the membership functions is zero, the min operator produces 0.

5.5 Aggregation (Conflict Resolution)

The *application of a control rule* is also called *firing*. *Aggregation* or *conflict resolution* is the methodology which is used in deciding what control action should be taken as a result of the firing of several rules.

Table 5.3 shows that only four rules have to be fired. The rest will not produce any results.

We will illustrate the process of conflict resolution by using those four rules numbered for convenience from one to four; they form a subset of (5.4):

Rule 1: *If* x *is* $\mathcal{A}_i^{(0)}$ *and* y *is* $\mathcal{B}_j^{(0)}$ *then* z *is* $\mathcal{C}_{ij}$,

Rule 2: *If* x *is* $\mathcal{A}_i^{(0)}$ *and* y *is* $\mathcal{B}_{j+1}^{(0)}$ *then* z *is* $\mathcal{C}_{i,j+1}$,

Rule 3: *If* x *is* $\mathcal{A}_{i+1}^{(0)}$ *and* y *is* $\mathcal{B}_j^{(0)}$ *then* z *is* $\mathcal{C}_{i+1,j}$,

Rule 4: *If* x *is* $\mathcal{A}_{i+1}^{(0)}$ *and* y *is* $\mathcal{B}_{j+1}^{(0)}$ *then* z *is* $\mathcal{C}_{i+1,j+1}$,

The *and* part of each rule, i.e. the *precondition*, called here *strength of the rule* or *level of firing* is denoted by

$$\begin{aligned}
\alpha_{ij} &= \mu_{\mathcal{A}_i}(x_0) \wedge \mu_{\mathcal{B}_j}(y_0) = \min(\mu_{\mathcal{A}_i}(x_0), \mu_{\mathcal{B}_j}(y_0)),\\
\alpha_{i,j+1} &= \mu_{\mathcal{A}_i}(x_0) \wedge \mu_{\mathcal{B}_{j+1}}(y_0) = \min(\mu_{\mathcal{A}_i}(x_0), \mu_{\mathcal{B}_{j+1}}(y_0)),\\
\alpha_{i+1,j} &= \mu_{\mathcal{A}_{i+1}}(x_0) \wedge \mu_{\mathcal{B}_j}(y_0) = \min(\mu_{\mathcal{A}_{i+1}}(x_0), \mu_{\mathcal{B}_j}(y_0)),\\
\alpha_{i+1,j+1} &= \mu_{\mathcal{A}_{i+1}}(x_0) \wedge \mu_{\mathcal{B}_{j+1}}(y_0) = \min(\mu_{\mathcal{A}_{i+1}}(x_0), \mu_{\mathcal{B}_{j+1}}(y_0)).
\end{aligned} \tag{5.10}$$

The equalities (5.10) can be obtained from (5.8) for $x = x_0$ and $y = y_0$. The real numbers $\alpha_{ij}, \alpha_{i,j+1}, \alpha_{i+1,j}$, and $\alpha_{i+1,j+1}$ are placed in the Table 5.4 called here *rules strength table.*

Table 5.4. Rules strength table.

	0	$\cdots$	$\mu_{\mathcal{B}_j}(y_0)$	$\mu_{\mathcal{B}_{j+1}}(y_0)$	$\cdots$	0
0	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\mu_{\mathcal{A}_i}(x_0)$	0	$\cdots$	α_{ij}	$\alpha_{i,j+1}$	$\cdots$	0
$\mu_{\mathcal{A}_{i+1}}(x_0$	0	$\cdots$	$\alpha_{i+1,j}$	$\alpha_{i+1,j+1}$	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
0	0	$\cdots$	0	0	$\cdots$	0

Table 5.4 is very similar to Table 5.3 with the difference that the active cells in Table 5.4 are occupied by the members expressing the strength of the rules while the same cells in Table 5.3 are occupied by fuzzy sets (outputs). We use the elements in the four active cells in both tables to introduce the notion *control output*.

Control output (CO) of each rule is defined by operation conjunction applied on its *strength* and *conclusion* as follows:

$$\begin{aligned}
&\text{CO of rule 1}: && \alpha_{ij} \wedge \mu_{\mathcal{C}_{ij}}(z) = \min(\alpha_{ij}, \mu_{\mathcal{C}_{ij}}(z)),\\
&\text{CO of rule 2}: && \alpha_{i,j+1} \wedge \mu_{\mathcal{C}_{i,j+1}}(z) = \min(\alpha_{i,j+1}, \mu_{\mathcal{C}_{i,j+1}}(z)),\\
&\text{CO of rule 3}: && \alpha_{i+1,j} \wedge \mu_{\mathcal{C}_{i+1,j}}(z) = \min(\alpha_{i+1,j}, \mu_{\mathcal{C}_{i+1,j}}(z)), \qquad (5.11)\\
&\text{CO of rule 4}: && \alpha_{i+1,j+1} \wedge \mu_{\mathcal{C}_{i+1,j+1}}(z) = \min(\alpha_{i+1,j+1}, \mu_{\mathcal{C}_{i+1,j+1}}(z)).
\end{aligned}$$

These control outputs can be obtained from (5.9) for $x = x_0, y = y_0$. This is equivalent to performing operation conjunction or min on the corresponding elements in the active cells in Table 5.4 and Table 5.3 as shown below

Table 5.5. Control outputs of rules 1–4.

$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\alpha_{ij} \wedge \mu_{\mathcal{C}_{ij}}(z)$	$\alpha_{i,j+1} \wedge \mu_{\mathcal{C}_{i,j+1}}(z)$	$\cdots$
$\cdots$	$\alpha_{i+1,j} \wedge \mu_{\mathcal{C}_{i+1,j}}(z)$	$\alpha_{i+1,j+1} \wedge \mu_{\mathcal{C}_{i+1,j+1}}(z)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

The nonactive cells have elements zero; they are not presented in Table 5.5.

The outputs of the four rules (5.11) located in the active cells (Table 5.5) now have to be *combined* or *aggregated* in order to produce one control output with membership function $\mu_{agg}(z)$. It is natural to use for aggregation the operator $\vee$ (*or*) expressed by max:

$$\begin{aligned}
\mu_{agg}(z) &= (\alpha_{ij} \wedge \mu_{\mathcal{C}_{ij}}(z)) \vee (\alpha_{i,j+1} \wedge \mu_{\mathcal{C}_{i,j+1}}(z))\\
&\quad \vee(\alpha_{i+1,j} \wedge \mu_{\mathcal{C}_{i+1,j}}(z)) \vee (\alpha_{i+1,j+1} \wedge \mu_{\mathcal{C}_{i+1,j+1}}(z))\\
&= \max\{(\alpha_{ij} \wedge \mu_{\mathcal{C}_{ij}}(z)), (\alpha_{i,j+1} \wedge \mu_{\mathcal{C}_{i,j+1}}(z)),\\
&\quad (\alpha_{i+1,j} \wedge \mu_{\mathcal{C}_{i+1,j}}(z)), (\alpha_{i+1,j+1} \wedge \mu_{\mathcal{C}_{i+1,j+1}}(z))\}. \qquad (5.12)
\end{aligned}$$

Note that in (5.11) and (5.12) operation $\wedge$ (min) is performed on a number and a membership function of a fuzzy set. Previously we have been using operation min on two numbers, two crisp sets, and two fuzzy sets, hence now some clarification is needed. Suppose we have the real number α and the fuzzy set $\mathcal{C}$ with membership function $\mu_{\mathcal{C}}(z)$. Then we define

$$\mu_{\alpha \wedge \mu_{\mathcal{C}}}(z) = \alpha \wedge \mu_{\mathcal{C}}(z) = \min(\mu_\alpha(z) = \alpha, \mu_{\mathcal{C}}(z)) \tag{5.13}$$

where $\mu_\alpha(z) = \alpha$ is a straight line parallel to z-axis; geometrically this is a truncation of the shape of $\mu_{\mathcal{C}}(z)$.

The membership function (5.13) is shown in Fig. 5.7 for the two most often used shapes of $\mu_{\mathcal{C}}(z)$, triangular and trapezoidal; it represents a clipped fuzzy number (a nonnormalized fuzzy set).

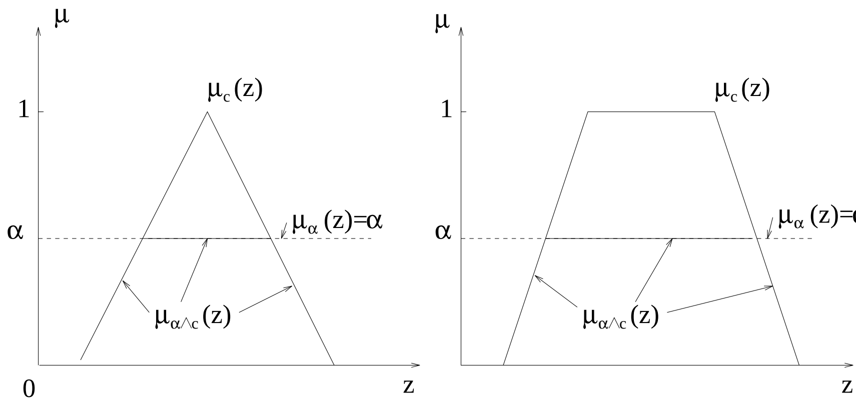

Fig. 5.7. Clipped triangular and trapezoidal numbers.

The aggregated membership function (5.12) also represents a nonnormalized fuzzy set consisting of parts of clipped membership functions (5.13) of the type shown on Fig. 5.7 (or similar). In order to obtain a crisp control output action, decision, or command we have to defuzzify $\mu_{agg}(z)$; this is the subject of the next section.

Case Study 17 (Part 3) *A Client Financial Risk Tolerance Model*

Consider Case Study 17 (Parts 1 and 2) assuming readings: $x_0 = 40$ in thousands (annual income) and $y_0 = 25$ in ten of thousands (total networth). They are matched against the appropriate terms in Fig. 5.8 (for the terms see Figs. 5.2 and 5.3). The fuzzy inputs are calculated from (5.3). Note that $x = 40$ and $y = 25$ are substituted for v instead of 40,000 and 250,000 since x and y are measured in thousands and ten of thousands. The result is

$$\mu_{\mathbf{L}}(40) = \frac{1}{3}, \quad \mu_{\mathbf{M}}(40) = \frac{2}{3}, \quad \mu_{\mathbf{L}}(25) = \frac{5}{6}, \quad \mu_{\mathbf{M}}(25) = \frac{1}{6}.$$

For $x = x_0 = 40$ and $y = y_0 = 25$ the decision Table 5.2 (a particular case of Table 5.1) reduces to the induced Table 5.6 (a particular case of Table 5.3).

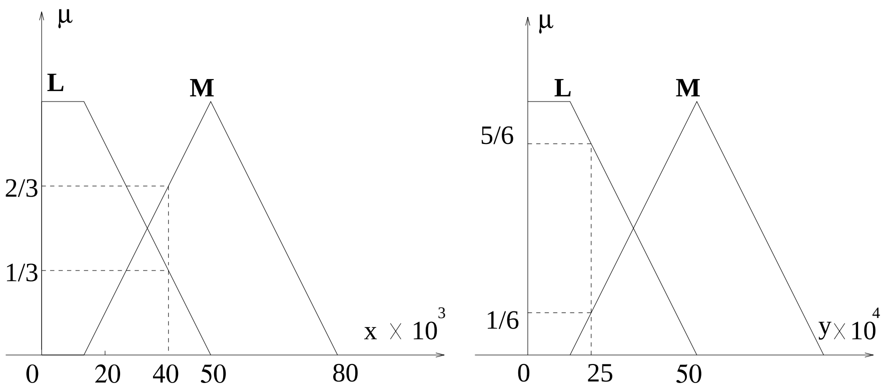

Fig. 5.8. Fuzzy reading inputs for the clients financial risk tolerance model. Readings: $x_0 = 40$ and $y_0 = 25$.

Table 5.6. Induced decision table for the clients financial risk tolerance model.

	$\mu_{\mathbf{L}}(25) = \frac{5}{6}$	$\mu_{\mathbf{M}}(25) = \frac{1}{6}$	0
$\mu_{\mathbf{L}}(40) = \frac{1}{3}$	$\mu_{\mathbf{L}}(z)$	$\mu_{\mathbf{L}}(z)$	0
$\mu_{\mathbf{M}}(40) = \frac{2}{3}$	$\mu_{\mathbf{L}}(z)$	$\mu_{\mathbf{MO}}(z)$	0
0	0	0	0

There are four active rules, 1,2,4,5 given in Case Study 17 (Part 2).

The strength of these rules (the *and* part) according to (5.10) is calculated as follows:

$$\begin{aligned} \alpha_{11} &= \mu_{\mathbf{L}}(40) \wedge \mu_{\mathbf{L}}(25) = \min(\tfrac{1}{3}, \tfrac{5}{6}) = \tfrac{1}{3}, \\ \alpha_{12} &= \mu_{\mathbf{L}}(40) \wedge \mu_{\mathbf{M}}(25) = \min(\tfrac{1}{3}, \tfrac{1}{6}) = \tfrac{1}{6}, \\ \alpha_{21} &= \mu_{\mathbf{M}}(40) \wedge \mu_{\mathbf{L}}(25) = \min(\tfrac{2}{3}, \tfrac{5}{6}) = \tfrac{2}{3}, \\ \alpha_{22} &= \mu_{\mathbf{M}}(40) \wedge \mu_{\mathbf{M}}(25) = \min(\tfrac{2}{3}, \tfrac{1}{6}) = \tfrac{1}{6}. \end{aligned} \tag{5.14}$$

These results are presented in the rules strength Table 5.7, a particular case of Table 5.4.

Table 5.7. Rules strength table for the clients financial risk tolerance model.

	$\mu_{\mathbf{L}}(25) = \frac{5}{6}$	$\mu_{\mathbf{M}}(25) = \frac{1}{6}$	0
$\mu_{\mathbf{L}}(40) = \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$\mu_{\mathbf{M}}(40) = \frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	0
0	0	0	0

For the control outputs (CO) of the rules we obtain from (5.11) with (5.14)

$$\begin{aligned} &\text{CO of rule 1} : \alpha_{11} \wedge \mu_{\mathbf{L}}(z) = \min(\tfrac{1}{3}, \mu_{\mathbf{L}}(z)), \\ &\text{CO of rule 2} : \alpha_{12} \wedge \mu_{\mathbf{L}}(z) = \min(\tfrac{1}{6}, \mu_{\mathbf{L}}(z)), \\ &\text{CO of rule 3} : \alpha_{21} \wedge \mu_{\mathbf{L}}(z) = \min(\tfrac{2}{3}, \mu_{\mathbf{L}}(z)), \\ &\text{CO of rule 4} : \alpha_{22} \wedge \mu_{\mathbf{MO}}(z) = \min(\tfrac{1}{6}, \mu_{\mathbf{MO}}(z)), \end{aligned} \tag{5.15}$$

which is equivalent to performing operation min on the corresponding cells in Table 5.7 and Table 5.6. The result concerning only the active cells (a particular case of Table 5.5) is given on Table 5.8.

Table 5.8. Control outputs for the client financial risk tolerance model.

...	...	...	...
...	$\frac{1}{3} \wedge \mu_{\mathbf{L}}(z)$	$\frac{1}{6} \wedge \mu_{\mathbf{L}}(z)$	...
...	$\frac{2}{3} \wedge \mu_{\mathbf{L}}(z)$	$\frac{1}{6} \wedge \mu_{\mathbf{MO}}(z)$	...
...	...	...	...

The procedure for obtaining Table 5.8 can be summarized on the scheme in Fig. 5.9 which consists of 12 triangular and trapezoidal fuzzy numbers located in 4 rows and 3 columns.

The min operations in (5.14) between the fuzzy inputs located in the first two columns (Fig. 5.9) produce correspondingly the strength of the rules $\frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{1}{6}$ which give the level of firing shown by dashed horizontal arrows in the second column in the direction to the triangles and trapezoidals in the third column.

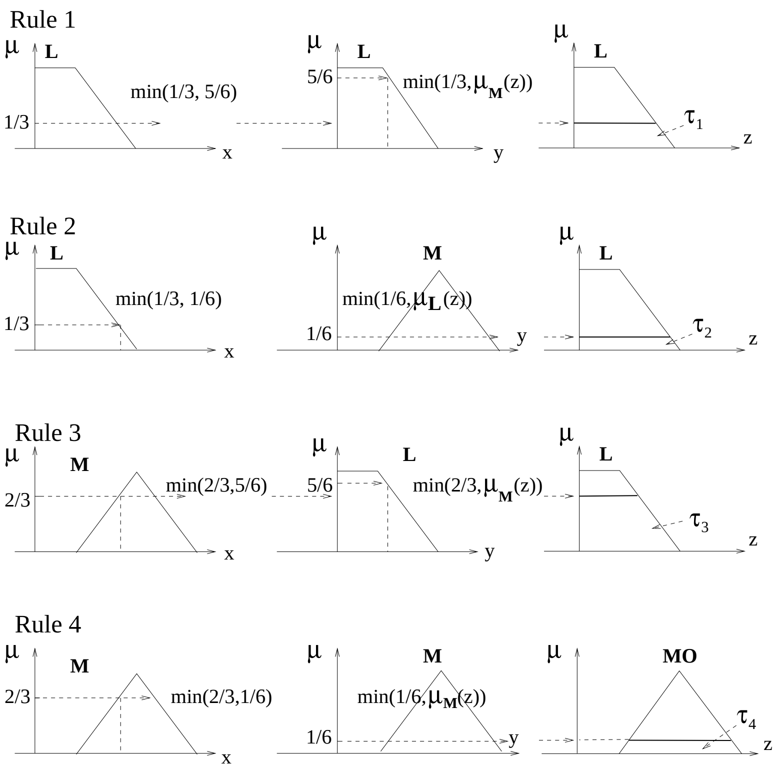

Fig. 5.9. Firing of rules for the client financial risk tolerance model.

The min operations in (5.15) in the sense of (5.13) and Fig. 5.7 result in the sliced triangular and trapezoidal numbers by the arrows (Fig. 5.9) thus producing the trapezoids $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, and $\mathcal{T}_4$.

To aggregate the control outputs (5.15) presented also on Table 5.8 we use (5.12). Geometrically this means that we have to superimpose trapezoids on top of one another in the same coordinate system (z, μ). However, the outputs of rule 1 and rule 2 are included in the output of rule 3 which has the largest strength $\frac{2}{3}$. This is shown in Fig. 5.9; the trapezoids $\mathcal{T}_1$ and $\mathcal{T}_2$ are contained in $\mathcal{T}_3$. Hence we may only consider aggregation of rule 3 and rule 4.

The aggregated output

$$\mu_{agg}(z) = \max\{\min(\frac{2}{3}, \mu_{\mathbf{L}}(z)), \min(\frac{1}{6}, \mu_{\mathbf{MO}}(z))\} \tag{5.16}$$

is geometrically presented in Fig. 5.10. The trapezoids $\mathcal{T}_3$ and $\mathcal{T}_4$ in Fig. 5.9 are superimposed a top one another.

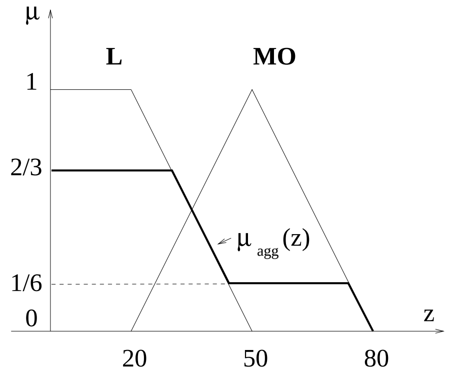

Fig. 5.10. Aggregated output for the client financial risk tolerance model.

□

5.6 Defuzzification

Defuzzification for average triangular and trapezoidal numbers was presented in Chapter 3, Section 3.3 and for a fuzzy set representing decision in Chapter 4, Section 4.1. Here we deal with a more complicated type of defuzzification.

Defuzzification or decoding the outputs is operation that produces a nonfuzzy control action, a single value $\hat{z}$, that adequately represents the membership function $\mu_{agg}(z)$ of an aggregated fuzzy control action.

There is no unique way to perform the operation defuzzification. The several existing methods for defuzzification[3] take into consideration the shape of the clipped fuzzy numbers, namely length of supporting intervals, height of the clipped triangles and trapezoids, closeness to central triangular numbers, and also complexity of computations.

We describe here three methods for defuzzification.

Center of area method (*CAM*)

Suppose the aggregated control rules result in a membership function $\mu_{agg}(z), z \in [z_0, z_q]$, shown in Fig. 5.11.

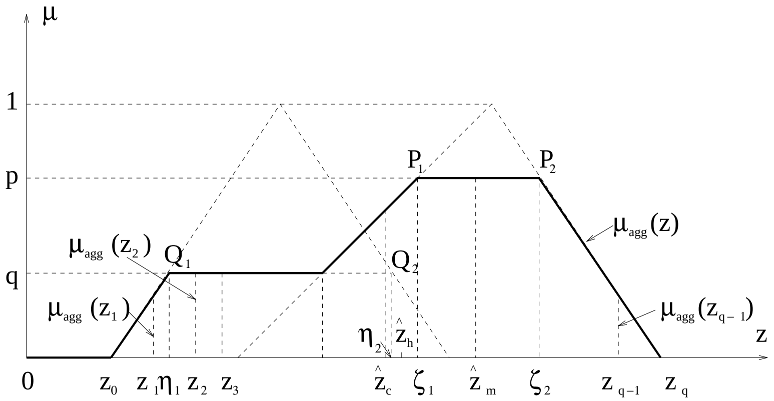

Fig. 5.11. Defuzzification by the center of area method (CAM).

Let us subdivide the interval $[z_0, z_q]$ into q equal (or almost equal) subintervals by the points $z_1, z_2, \ldots, z_{q-1}$.

The crisp value $\hat{z}_c$ according to this method is the weighted average of the numbers z_k (see (3.2) where now $r_k = z_k$ and $\lambda_k = \mu_{agg}(z_k)$),

$$\hat{z}_c = \frac{\sum_{k=1}^{q-1} z_k \mu_{agg}(z_k)}{\sum_{k=1}^{q-1} \mu_{agg}(z_k)}. \tag{5.17}$$

The geometric interpretation of $\hat{z}_c$ is that it is the first coordinate (abscissa) of the center $(\hat{z}_c, \mu_C)$ of the area under the curve $\mu_{agg}(z)$

bounded below by the z-axis. The physical interpretation is that if this area is cut off from a thin piece of metal or wood, the center of the area will be the center of gravity. That is why CAM is called also *center of gravity method.*

This method for defuzzification, perhaps the most popular, is quite natural from point of view of common sense. However, the required computations are sometimes complex.

Mean of maximum method (MMM)

Consider the same membership function $\mu_{agg}(z)$ as in the center of area method (Fig. 5.11). The function has two flat segments (parallel to z axis). The projection of the flat segment P_1P_2 with maximum height on z axis is the interval $[\zeta_1, \zeta_2]$ (see Fig. 5.11). Then neglecting the contribution of the clipped triangular number with flat segment Q_1Q_2 we define $\hat{z}_m$ to be the midpoint of the interval $[\zeta_1, \zeta_2]$, i.e.

$$\hat{z}_m = \frac{\zeta_1 + \zeta_2}{2}. \tag{5.18}$$

This is a simple formula but not very accurate.

Height defuzzification method (HDM)

This is a generalization of mean of maximum method. It uses all clipped flat segments obtained as result of firing rules (see Fig. 5.11). Besides the segment P_1P_2 with height p there is another flat segment Q_1Q_2 with lower height q. The midpoint of the interval $[\eta_1, \eta_2]$, the projection of Q_1Q_2 on z, is $\frac{\eta_1+\eta_2}{2}$. Then the HDM produces $\hat{z}_h$:

$$\hat{z}_h = \frac{p\frac{\zeta_1+\zeta_2}{2} + q\frac{\eta_1+\eta_2}{2}}{p+q} = w_1\frac{\zeta_1 + \zeta_2}{2} + w_2\frac{\eta_1 + \eta_2}{2}, \tag{5.19}$$

i.e. $\hat{z}_h$ is the weighted average (3.2) of the midpoints of $[\zeta_1, \zeta_2]$ and $[\eta_1, \eta_2]$ with weights $w_1 = \frac{p}{p+q}, w_2 = \frac{q}{p+q}$, where p and q are the heights of the flat segments.

If there are more than two segments, formula (5.19) can be extended accordingly.

HDM could be considered as both a simplified version of CAM and a generalized version of MMM.

Case Study 17 (Part 4) *A Client Financial Risk Tolerance Model*

Let us defuzzify the aggregated output for the client financial risk tolerance model (Case Study 17 (Part 3)) by the three methods.

First we express analytically the aggregated control output with membership function $\mu_{agg}(z)$ shown on Fig. 5.12 (see also (5.10)). It consists of the four segments P_1P_2, P_2Q, QQ_2, and Q_2Q_3 located on the straight lines $\mu = \frac{2}{3}$, $\mu = \frac{50-z}{30}$, $\mu = \frac{1}{6}$, and $\mu = \frac{80-z}{30}$, correspondingly. Solving together the appropriate equations gives the projections of P_2, Q, Q_2 on z axis, namely 30, 45, 75 (Fig. 5.12). They are used to specify the domains of the segments forming $\mu_{agg}(z)$. Hence

$$\mu_{agg}(z) = \begin{cases} \frac{2}{3} & \text{for} \quad 0 \leq z \leq 30, \\ \frac{50-z}{30} & \text{for} \quad 30 \leq z < 45, \\ \frac{1}{6} & \text{for} \quad 45 \leq z < 75, \\ \frac{-z+80}{30} & \text{for} \quad 75 \leq z \leq 80. \end{cases}$$

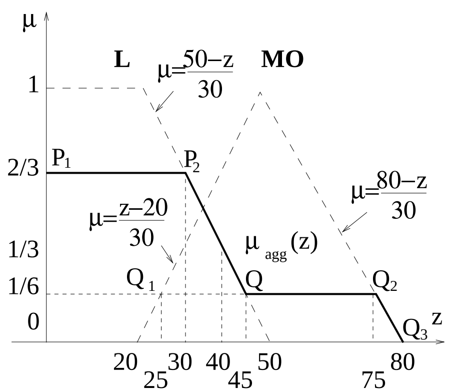

Fig. 5.12. Defuzzification: client financial risk tolerance model.

Center of area method

It is convenient to subdivide the interval [0,80] (Fig. 5.12) into eight equal parts each with length 10.

The substitution of $z_k = 10, 20, \ldots, 70$ into $\mu_{agg}(z)$ gives

z_k	10	20	30	40	50	60	70
$\mu_{agg}(z_k)$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

According to (5.17) we find,

$$\begin{aligned}\hat{z}_c &= \frac{10(\frac{2}{3}) + 20(\frac{2}{3}) + 30(\frac{2}{3}) + 40(\frac{1}{3}) + 50(\frac{1}{6}) + 60(\frac{1}{6}) + 70(\frac{1}{6})}{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}} \\ &= 29.41\,.\end{aligned}$$

Mean of maximum method

The points P_1, P_2 form the highest flat segment, $\zeta_1 = 0$ and $\zeta_2 = 30$.
Then (5.18) gives

$$\hat{z}_m = \frac{0 + 30}{2} = 15.$$

Height defuzzification method

Substituting $\mu = \frac{1}{6}$ into $\mu = \frac{z-20}{30}$ gives the number 25, the projection of the point Q_1. Hence the flat segments P_1P_2 and Q_1Q_2 in Fig. 5.12 have projections [0,30] and [25, 75], and heights $\frac{2}{3}$ and $\frac{1}{6}$, correspondingly, i.e. $\zeta_1 = 0, \zeta_2 = 30, \eta_1 = 25, \eta_2 = 75, p = \frac{2}{3}, q = \frac{1}{6}$. The result of substituting these values in (5.19) is

$$\hat{z}_h = \frac{\frac{2}{3}\frac{0+30}{2} + \frac{1}{6}\frac{25+75}{2}}{\frac{2}{3} + \frac{1}{6}} = 22\,.$$

The defuzzification results $\hat{z}_c = 29.41 \approx 29, \hat{z}_m = 15$, and $\hat{z}_h = 22$ obtained by the three methods are close. MMM is very easy to apply but produces here an underestimated result since it neglects the contribution of rule 4 whose firing level $\frac{1}{6}$ intersects the output **MO**; $\hat{z}_m$ lies in the middle of the supporting interval of output **L**. CAM requires some calculations but takes into consideration the contributions of both rules, 3 and 4. The value $\hat{z}_c$ looks more realistic than $\hat{z}_m$. The HDM results in a value $\hat{z}_h = 22$; it is easy to apply and similarly to CAM reflects the contributions of rules 3 and 4.

The financial experts could estimate the clients financial risk tolerance given that his/her annual income is 40,000 and total networth is 250,000 to be 22 on a scale from 0 to 100 if they adopt the HDM (29 if they adopt CAM). Accordingly they could suggest a conservative risk investment strategy.

□

5.7 Use of Singletons to Model Outputs

A segment or interval $[0, h], h \leq 1$ is its height, parallel to the vertical axis μ is considered as a fuzzy singleton (see Section 1.2).

Aggregation procedure and defuzzification calculations can be carried out more easily in comparison to those introduced in Sections 5.5 and 5.6 if singletons with height one are chosen to represent the terms $\mathcal{C}_k$ (see 5.2) of the output $\mathcal{C}$ (see 5.1).

This is illustrated on the client financial risk tolerance model (Case Study 17 (Parts 1–4)).

Case Study 18 *Use of Singletons for a Client Financial Risk Tolerance Model*

Assume that the financial experts use singletons to model the output risk tolerance (see Fig. 5.13(a)) while the inputs are defined as in Case Study 17 (Part 1). Hence instead of the three fuzzy numbers in Fig. 5.4 now there are three singletons in Fig. 5.13(a).

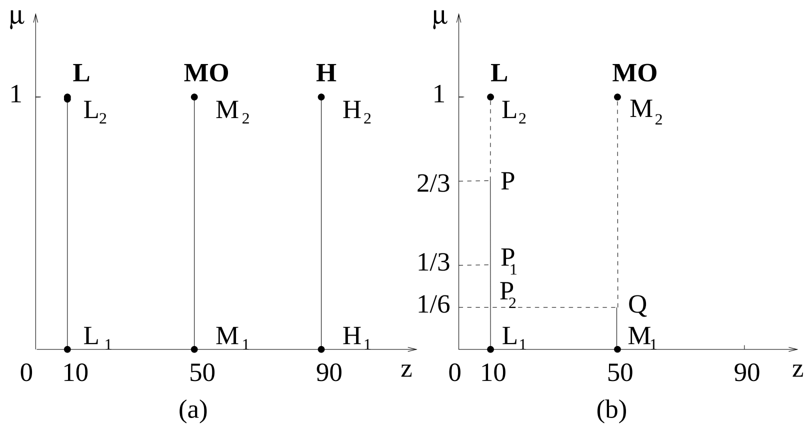

Fig. 5.13. (a) Terms of the output *risk tolerance* presented by singletons. (b) Firing of rules and defuzzification.

Consider the same *if* ... *and* ... *then* rules given in Table 5.2. Now **L**, **MO**, and **H** are singletons, not triangular and trapezoidal numbers. Also adopt the same readings as in Case Study 17 (Part 3) shown on Fig. 5.8. Then formula (5.14) expressing the strength of the rules is valid. The control outputs (5.15) are valid but now $\mu_{\mathbf{L}}(z)$ and $\mu_{\mathbf{MO}}(z)$ are substituted by the singletons **L** and **MO** shown in Fig. 5.13 (a).

The firing of the rules follows the procedure schematically presented in Fig. 5.9. The first two columns of figures remain without change. There is a difference only in the third column—the terms **L**, **L**, **L**, and **MO** are substituted by the corresponding singletons.

The min operations (5.15) expressing the control outputs now result in sliced singletons presented in one figure (Fig. 5.13 (b))—not in four as in Fig. 5.9. The firing of rules 1 and 2 cut the segments L_1P_1 and L_1P_2 out from the singleton **L**. The firing of rule 3 cut the segment L_1P out from the singleton **L**; it includes the segments L_1P_1 and L_1P_2. The firing of rule 4 cut the segment M_1Q out from the singleton **MO**. Hence only two segments, L_1P and M_1Q form the aggregated output (Fig. 5.13 (b)).

Operation defuzzification is performed by calculating the weighted average (see (3.2)) of the points L_1 and M_1 representing 10 and 50:

$$\hat{z} = \frac{\frac{2}{3}(10) + \frac{1}{6}(50)}{\frac{1}{6} + \frac{2}{3}} = 18.$$

Essentially this is a particular case of formula (5.17), CAM, and also particular case of (5.19), HDM.

The resulting number 18 is more conservative than 29 and 22 produced correspondingly by CAM and HDM when the terms of the output $\mathcal{C}$ were described not by singletons but by fuzzy numbers (see Case Study 17 (Part 4)).

□

When using singletons, we can expect results close (or equal) to those which we could get by using fuzzy terms, but not better. Advantage: simplified calculations. Disadvantage: disconnected segment outputs (see Fig. 5.13 (b)) weakened the protection of partly overlapping fuzzy outputs against a model which might be good to lesser degree.

5.8 Tuning of Fuzzy Logic Control Models

In Section 5.2 four steps for designing the terms $\mathcal{A}_i, \mathcal{B}_j$, and $\mathcal{C}_k$ (see (5.2)) have been presented. In Section 5.3 *if ... then* rules involving these terms (see (5.4)) have been formally constructed. That, together with the readings, predetermines the final result obtained by applying FLC. However in some situations the experts may find the results to be somewhat not very satisfactory from common-sense point of view and this may raise doubt in their own judgement. Then the experts have the option to improve the FLC model by modification and revision of the shapes and number of terms, location of peaks, flats, supporting intervals. Also they may reconsider and redesign the control rules. This revision is called *tuning* or *refinement*. Unfortunately there is no unique method for such tuning. There are some suggestions in the engineering literature but this is out of the scope of the book. The experts who designed the FLC model using their good knowledge and experience would simply have to do more work and thinking to improve the model if they feel that this may bring better results.

As an illustration again we use the model in Case Study 17 (Parts 1–4).

Case Study 19 *Tuning of a Client Financial Risk Tolerance Model*

Assume the experts consider the conclusion of the FLC model, namely the crisp value 22(HDM) measuring the risk tolerance on the scale from 0 to 100 to be too small for a person with annual income 40,000 and total networth 250,000. Hence they decide to tune the model making slight change to the terms of output $\mathcal{C}$-*risk tolerance*. The modified terms are shown on Fig. 5.14.

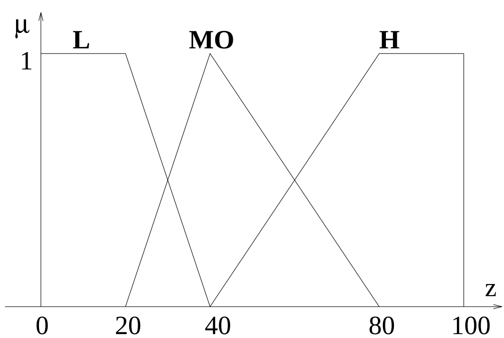

Fig. 5.14. Modified terms of the output *risk tolerance*.

In comparison to Fig. 5.4 there are several changes: (1) The new terms **L** and **H** have new supporting intervals [0, 40] instead of [0, 50] and [40, 100] instead of [50, 100], correspondingly; (2) the new term **MO** has its peak shifted to the left by 10 units; it is still a triangular number but not in central form.

Assuming everything else in the model in Case Study 17 (Parts 1–4) stays without change, firing of the same rules produces here the aggregated output given in Fig. 5.15.

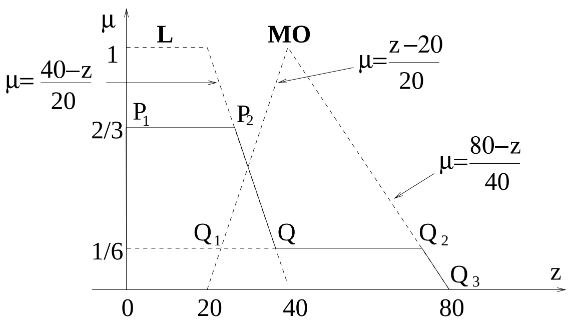

Fig. 5.15. Aggregated outputs and defuzzification for the tuned client financial risk tolerance model.

Solving together $\mu = \frac{2}{3}$ and $\mu = \frac{40-z}{20}$, $\mu = \frac{1}{6}$ and $\mu = \frac{z-20}{20}$, $\mu = \frac{1}{6}$ and $\mu = \frac{80-z}{40}$ we find that the projections of P_1P_2 and Q_1Q_2 are $[0, \frac{80}{3}]$ and $[\frac{70}{3}, \frac{220}{3}]$.

The HDM (formula (5.19)) gives the nonfuzzy control output

$$\hat{z}_h^t = \frac{\frac{2}{3}\frac{0+\frac{80}{3}}{2} + \frac{1}{6}\frac{\frac{70}{3}+\frac{220}{3}}{3}}{\frac{2}{3}+\frac{1}{6}} = 30.$$

This value is larger than 22 of the initial model obtained by HDM. It suggests a quite moderate financial risk tolerance.

□

5.9 One-Input–One-Output Control Model

It was noted in the beginning of Section 5.2 that the control methodology can be applied to the simple case of one-input–one-output.

Let us consider as an illustration one input $\mathcal{A}$ and one output $\mathcal{C}$ each consisting of four terms of triangular shape (see Figs. 5.16 and 5.17).

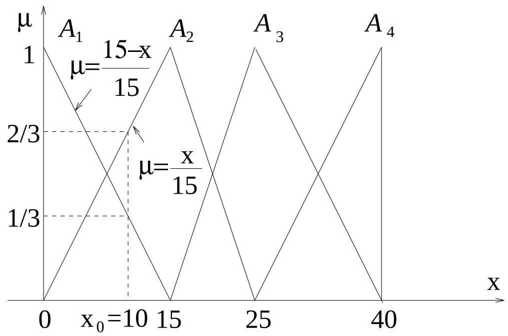

Fig. 5.16. Input $\mathcal{A}$; terms of $\mathcal{A}$. Reading x_0 and fuzzy reading inputs.

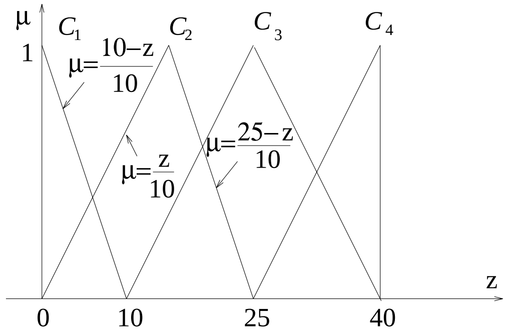

Fig. 5.17. Output $\mathcal{C}$; terms of $\mathcal{C}$.

The number of the *if* ... *then* rules is four – that is the number of terms in the input $\mathcal{A}$. Since there is no second input, the rules do not contain the *and* connective; they are of the type (5.4) but *and* and $\mathcal{B}_j$ are missing.

Assume the rules are

Rule 1: *If* x *is* $\mathcal{A}_1$ *then* $\mathcal{C}_1$,

Rule 2: *If* x *is* $\mathcal{A}_2$ *then* $\mathcal{C}_2$,

Rule 3: *If* x *is* $\mathcal{A}_3$ *then* $\mathcal{C}_3$,

Rule 4: *If* x *is* $\mathcal{A}_4$ *then* $\mathcal{C}_4$,

It is not necessary for $\mathcal{C}_i$ to take part in rule $i, i = 1, \ldots, 4$. That depends on the meaning of $\mathcal{A}_i$ and $\mathcal{C}_i$ in a particular situation.

Assume reading $x_0 = 10$. Then substituting 10 for x into $\mu = \frac{15-x}{15}$ and $\mu = \frac{x}{15}$ gives the fuzzy reading inputs $\frac{1}{3}$ and $\frac{2}{3}$ (see Fig. 5.16).

Since there is only one input, the strengths or the rules or levels of firing (5.10) reduce to $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$, hence two rules are to be fired.

The control output (CO) of each rule (see 5.11) is

CO of rule 1: $\alpha_1 \wedge \mu_{\mathcal{C}_1}(z) = \min(\frac{1}{3}, \mu_{\mathcal{C}_1}(z))$,

CO of rule 2: $\alpha_2 \wedge \mu_{\mathcal{C}_2}(z) = \min(\frac{2}{3}, \mu_{\mathcal{C}_1}(z))$.

The firing of these rules produces independently two clipped triangular numbers. The operation is presented in one figure (Fig. 5.18).

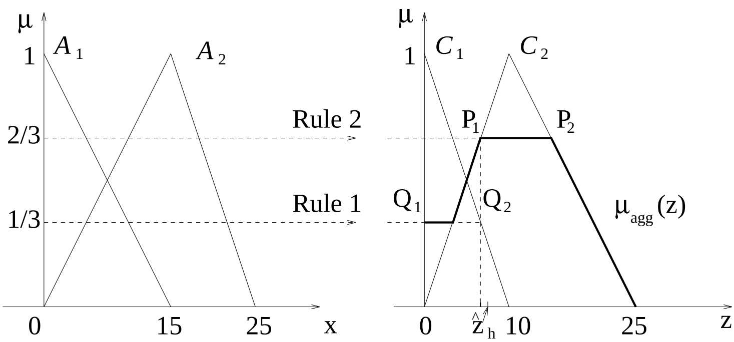

Fig. 5.18. Firing of two rules. Aggregated output $\mu_{agg}(z)$.

The sliced triangular numbers $\mathcal{C}_1$ and $\mathcal{C}_2$ give two trapezoids whose aggregated output is $\mu_{agg}(z)$ shown on Fig. 5.18 with tick lines,

$$\mu_{agg}(z) = \max(\min(\frac{1}{3}, \mu_{\mathcal{C}_1}(z)), \min(\frac{2}{3}, \mu_{\mathcal{C}_2}(z));$$

it is a particular case of (5.12).

For defuzzification we apply the HDM. Substituting $\mu = \frac{1}{3}$ into $\mu = \frac{10-z}{10}$ and $\mu = \frac{2}{3}$ into $\mu = \frac{z}{10}$ and into $\mu = \frac{25-z}{15}$ gives the numbers $\frac{20}{3}, \frac{20}{3}, 15$, hence the projections of P_1P_2 and Q_1Q_2 are $[\frac{20}{3}, 15]$ and $[0, \frac{20}{3}]$.

Using formula (5.19) we obtain

$$\hat{z}_h = \frac{\frac{2}{3}\frac{\frac{20}{3}+15}{2} + \frac{1}{3}\frac{0+\frac{20}{3}}{2}}{\frac{2}{3}+\frac{1}{3}} = 8.33.$$

5.10 Notes

1. The conceptual base for fuzzy logic control was established by Zadeh (1973) in the paper *Outline of a New Approach to the Analysis of Complex Systems and Decision Processes.* Zadeh's paper inspired Mamdani to introduce a specific fuzzy control methodology (Mamdani and Assilian (1975)) which was later developed further, extended, and applied by many researchers to different industrial engineering problems. A modern monograph book on fuzzy modeling and control has been written by Yager and Filev (1994).

2. Consider more than two inputs (but one output), say three having correspondingly n, m, and p terms. Then the inference rules will be of the type *if* ... *and* ... *and* ... *then* involving two logical connectives *and.* The number of the rules is determined by the product $n \times m \times p$. Accordingly this can be generalized for more inputs. For instance, if $n = m = p = 3$, the number of rules is $3 \times 3 \times 3 = 3^3 = 27$. If another (fourth) input also with three terms is added, the number of rules becomes $27 \times 3 = 3^4 = 81$, etc. Naturally more than two inputs will cause difficulties and they will increase faster than the increase of the number of inputs. The use of computer programs helps. In Chapter 6, Section 6.4, a simplified FLC technique is used in a case with three inputs. Also it is possible to have models with more than one output. The number of outputs requires the same number of decision tables. A two-input–three-output FLC models is presented in Chapter 6, Section 6.1.

3. Six defuzzification methods are described and analyzed by Hellendoorn and Thomas (1993).